stat 84 lec 17

warmup

What is $\sum_{k=0}^{\infty} p^{i} = 1+p+p^2+\dots = \frac{1}{1-p}$, $0<p<1$

$\frac{d}{dp}$ $\quad$ $\frac{d}{dp}$

$\sum_{k=0}^{\infty} kp^{k-1}$, $0<p<1$ $\quad \frac{1}{(1-p)^2}$

$$\Rightarrow \quad \sum_{k=0}^{\infty} kp^{k-1} = \frac{1}{(1-p)^2}$$

Arjun Venkatesh

2:39pm

Section 3.4, problem 10, parts a, b, and c from the textbook: "Let X be the number of Bernoulli (p) trials required to produce at least one success and at least one failure." Thanks!

Reply

10. Let X be the number of Bernoulli (p) trials required to produce at least one success and at least one failure. Find:

a) the distribution of X; b) $E(X)$; c) $Var(X)$.

Helpful identities:

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$$ geometric sum

$$\frac{d}{dq}$$

$$\sum_{k=0}^{\infty} k q^{k-1} = \frac{1}{(1-q)^2}$$ ⟵ useful for part b

$$\frac{d}{dq}$$

$$\boxed{\sum_{k=0}^{\infty} k(k-1) q^{k-2} = \frac{2}{(1-q)^3} = \frac{2}{p^3}}$$

Claim $$\sum_{k=1}^{\infty} k^2 q^{k-1} = \frac{1+q}{(1-q)^3}$$ ⟵ useful for part c

Pf/ let $C = \sum_{k=1}^{\infty} k(k-1) q^{k-2} = \sum_{k=1}^{\infty} k^2 q^{k-2} - \sum_{k=1}^{\infty} k q^{k-2} = \frac{2}{(1-q)^3}$

then $qC = \sum_{k=1}^{\infty} k^2 q^{k-1} - \sum_{k=1}^{\infty} k q^{k-1}$

$$\Rightarrow \sum_{k=1}^{\infty} k^2 q^{k-1} = qC + \sum_{k=1}^{\infty} k q^{k-1}$$

$$= \frac{2q}{(1-q)^3} + \frac{1}{(1-q)^2} = \frac{1+q}{(1-q)^3}$$

a) $P(X=1)=0$

$P(X=2)=qp+pq$

$P(X=3)=q^2p+p^2q$

$\vdots$

$P(X=k)=q^{k-1}p+p^{k-1}q\,,\ k\geq 2$

b)

$$E(X)=\sum_{k=2}^{\infty}k\left(q^{k-1}p+p^{k-1}q\right)=p\sum_{k=2}^{\infty}kq^{k-1}+q\sum_{k=2}^{\infty}kp^{k-1}$$

$$=p\left[\sum_{k=1}^{\infty}kq^{k-1}-1\right]+q\left[\sum_{k=1}^{\infty}kp^{k-1}-1\right]$$

$$=p\left[\frac{1}{(1-q)^2}-1\right]+q\left[\frac{1}{(1-p)^2}-1\right]$$

$$=p\left[\frac{1}{p^2}-1\right]+q\left[\frac{1}{q^2}-1\right]$$

$$=\frac{1}{p}-p+\frac{1}{q}-q=\frac{1}{p}+\frac{1}{q}-1$$

c)

$$E(X^2)=\sum_{k=2}^{\infty}k^2\left(q^{k-1}p+p^{k-1}q\right)$$

$$=p\left(\sum_{k=1}^{\infty}k^2q^{k-1}-1\right)+q\left(\sum_{k=1}^{\infty}k^2p^{k-1}-1\right)$$

$$=p\left(\frac{1+q}{(1-q)^3}-1\right)+q\left(\frac{1+p}{(1-p)^3}-1\right)=\frac{1+p}{q^2}+\frac{1+q}{p^2}-1$$

Finally use $Var(X)=E(X^2)-(E(X))^2$

Jenny Gao

7:43pm

The last example problem in Lecture 16, "conditional distribution, Poisson".

Thanks!!

Edited by Jenny Gao on Feb 26 at 7:43pm

Reply

red cars crossing 66B in 1min

blue cars crossing 66B in 1min

~~ex~~ Conditional distribution, Poisson

8. Let X_1 and X_2 be independent random variables such that for $i = 1, 2$, the distribution of X_i is Poisson (μ_i). Let m be a fixed positive integer. Find the distribution of X_1 given that $X_1 + X_2 = m$. Recognize this distribution as one of the famous ones, and provide its name and parameters.

$$\left.\begin{array}{l} X_1 \sim \text{Pois}(\mu_1) \\ X_2 \sim \text{Pois}(\mu_2) \end{array}\right\} \text{indep} \Rightarrow X_1 + X_2 \sim \text{Pois}(\mu_1+\mu_2) \qquad P(X_1 = k) = \frac{e^{-\mu_1}\mu_1^k}{k!}$$

$X_1 \mid X_1 + X_2 = m$ takes values $0, 1, 2, \ldots, m$

$p = \frac{\mu_1}{\mu_1+\mu_2}$ Guess $\text{Bin}\left(m, \frac{\mu_1}{\mu_1+\mu_2}\right)$

$X_2 = m-k$

$$P(X_1 = k \mid X_1 + X_2 = m) = \frac{P(X_1 = k, X_1 + X_2 = m)}{P(X_1 + X_2 = m)}$$

$$= \frac{P(X_1 = k)P(X_2 = m-k)}{P(X_1 + X_2 = m)}$$

$$= \frac{\frac{\cancel{e^{-\mu_1}}\mu_1^k}{k!} \cdot \frac{\cancel{e^{-\mu_2}}\mu_2^{m-k}}{(m-k)!}}{\frac{\cancel{e^{-(\mu_1+\mu_2)}}(\mu_1+\mu_2)^m}{m!}} = \binom{m}{k}\left(\frac{\mu_1}{\mu_1+\mu_2}\right)^k \left(\frac{\mu_2}{\mu_1+\mu_2}\right)^{m-k}$$

$$\Rightarrow \boxed{X_1 \mid X_1 + X_2 = m \sim \text{Bin}\left(m, \frac{\mu_1}{\mu_1+\mu_2}\right)}$$

Yusra Elbgal

5:05pm

Lecture 14 coupon collector's problem, when identifying the distribution of the boxes where you said "If it was the number of trials until the second success, it would be a negative binomial. But this is just the number of trials after the first success. So x2 is just after the first successes, the number of trials until the next prize". How do we identify a negative binomial distribution versus geometric distribution?

Coupon Collector's problem

You have a collection of boxes each containing a coupon. There are n different coupons. Each box is equally likely to contain any coupon independent of the other boxes.

X = # boxes needed to get all n different coupons.

ex $n=3$ $\quad X = X_1 + X_2 + X_3$

$$\underbrace{c}_{X_1}\underbrace{cc}_{X_2}\underbrace{ccccc\,c}_{X_3}$$

$X_1 = \text{Geom}\left(\frac{3}{3}\right)$, $X_2 = \text{Geom}\left(\frac{2}{3}\right)$, $X_3 = \text{Geom}\left(\frac{1}{3}\right)$

a) What is the distribution of X_1, X_2, X_3?
Are they independent?

Negative Binomial Distribution (NegBin(r, p))

generalization of Geom(p)

Sum of r indep Geom(p) on $\{r, r+1, r+2, \ldots\}$

Compare coupon collector problem with
X = # trials until 5^{th} orange coupon.
$X \sim NB\left(5, \frac{1}{n}\right)$

Aryan Shafat

4:20pm

3.3.8- e- Variance question where the events are subsets of each other.

2.5.8- a,b,c- In a raffle with 100 tickets

2.5.12- a, e- Poker hands

2.rev.10- a, b- lie detector test question

2.rev.18- b, c, d, e- 7 dice are rolled.

8. Let A_1, A_2, and A_3 be events with probabilities $\frac{1}{5}$, $\frac{1}{4}$, and $\frac{1}{3}$, respectively. Let N be the number of these events that occur.

a) Write down a formula for N in terms of indicators.

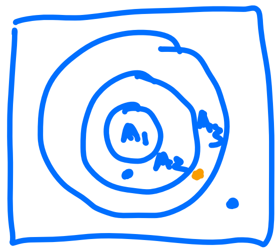
b) Find $E(N)$. $\leftarrow$ $E(N) = 1/5 + 1/4 + 1/3 = \boxed{\frac{47}{60}}$

In each of the following cases, calculate $Var(N)$:

c) A_1, A_2, A_3 are disjoint;

d) they are independent;

e) $A_1 \subset A_2 \subset A_3$.



$$Var(N) = E(N^2) - (E(N))^2$$

$$E(N^2) = 1^2 \cdot \frac{1}{12} + 2^2 \cdot \frac{1}{20} + 3^2 \cdot \frac{1}{5} = \frac{25}{12}$$

$$E(N) = 47/50$$

n	0	1	2	3
$P(N=n)$	$1 - 1/3 = 2/3$	$\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$	$\frac{1}{4} - \frac{1}{5}$ $\frac{1}{20}$	$\frac{1}{5}$

8. In a raffle with 100 tickets, 10 people buy 10 tickets each. If there are 3 winning tickets drawn at random find the probability that:

a) one person gets all 3 winning tickets;

b) there are 3 different winners;

c) some person gets two winners and someone else gets just one.

Think of the outcome space a 3 element subsets from the set $\{1, \ldots, 100\}$

a) $\binom{10}{1} \dfrac{\binom{10}{3}}{\binom{100}{3}}$

b) $\dfrac{\binom{10}{3}\binom{10}{1}\binom{10}{1}\binom{10}{1}}{\binom{100}{3}}$

c) $1 - a) - b)$

double single

or $= \dfrac{\binom{10}{1}\binom{9}{1}\binom{10}{2}\binom{10}{1}}{\binom{100}{3}}$

12. Poker hands. Assume all $\binom{52}{5}$ hands equally likely. Find the probability of being dealt:

a) a straight flush (5 consecutive cards of the same suit);

b) four of a kind (ranks a, a, a, a, b);

c) a full house (ranks a, a, a, b, b);

d) a flush (5 of the same suit, not a straight flush);

e) a straight (5 consecutive ranks, not a flush); straight

f) three of a kind (ranks a, a, a, b, c);

g) two pairs (ranks a, a, b, b, c);

h) a pair (ranks a, a, b, c, d);

i) none of the above.

a) lowest rank omit J,Q,K $\dfrac{\binom{10}{1}\binom{4}{1}}{\binom{52}{5}}$ suit

e) lowest rank $\dfrac{\binom{10}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}} - a)$

Katherine Kavounas
4:42pm

In order of priority:

1. Breakdown of problem types for decks of cards: when to use which techniques we have learned

2. Quiz 2, Question 1: "In a certain card game for a 52 card deck..."

3. Summary of Bayes' Rule / Forwards and Backwards Conditional Probabilities

4. What changes when we must find P(x > k), instead of P(X = k)

5. Refresher on when to use Chebyshev's vs. Markov's inequality to establish upper bounds

Quiz 2

In a certain card game for a 52 card deck, each card has a point value.

- Numbered cards in the range 2 to 9 are worth five points each.
- The cards numbered 10 and the face cards (jack, queen, king) are worth ten points each.
- Aces are worth fifteen points each.

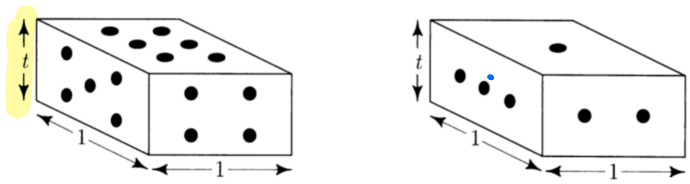
1. What is the expected number of numbered cards (in range 2 to 9) you get when you draw 3 cards?
2. We pick 3 cards at random. What is the expected total point value of the three cards on the top of the deck after the shuffle?

1. Let X_1 be the the number of numbered cards out of 3. $X_1 = I_1 + I_2 + I_3$ where I_2 is 1 if the second card is a numbered card. This has probability 8/13. It follows that $E(X_1) = 3(8/13) = 24/13$.
2. Let X_1 be the number of numbered cards in 3 draws, X_2 the number of face cards in 3 draws, and X_3 the number of aces in 3 draws. Let X be the total point value of the three cards. We have $X = 5X_1 + 10X_2 + 15X_3$ and want $E(X)$. Similar to part (a), $E(X_2) = 3(4/13) = 12/13$ and $E(X_3) = 3(1/13) = 3/13$. Then $E(X) = 5(24/13) + 10(12/13) + 15(3/13) = 288/13$.

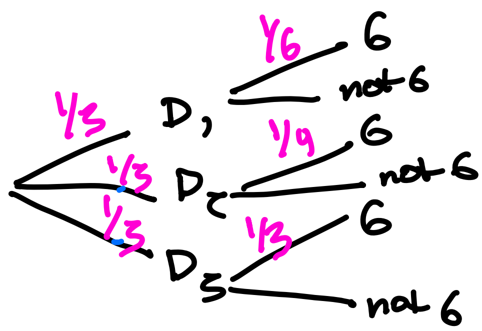
Forward conditional vs Backward conditional.

Shapes.

A *shape* is a 6-sided die with faces cut as shown in the following diagram:



A box contains 3 shaped die (see pic above), D_1, D_2, D_3, with probability $1/3$, $1/2$, $2/3$ respectively of landing flat (with 1 or 6 on top)



$P(\text{get } 6 \mid D_1) = 1/6$ forward conditional

$$P(D_1 \mid \text{get } 6) = \frac{P(D_1, \text{get } 6)}{P(\text{get } 6)} = \frac{\frac{1}{3}\cdot\frac{1}{6}}{\frac{1}{3}\cdot\frac{1}{6} + \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{3}\cdot\frac{1}{3}}$$

backward conditional