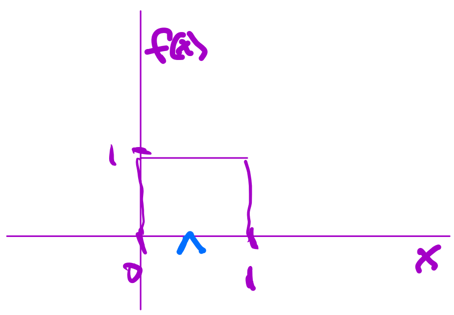
Stat 134 lec 19 (no lec 18)

warmup

Let $X \sim \text{Unif}(0,1)$ be the standard uniform distribution with histogram (density)

$$f(x) = \begin{cases} 1 & \text{if } 0 < x \le 1 \\ 0 & \text{else} \end{cases}$$

Picture



Define

$$E(X) = \int_{x=-\infty}^{\infty} x f(x)\,dx$$

Find $E(X)$, $E(X^2)$, and $Var(X)$.

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x f(x)\,dx = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \boxed{1/2}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \boxed{1/3}$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{1}{3} - \frac{1}{4} = \boxed{1/12}$$

Last time

Congratulations on finishing midterm 1 !

today

Sec 4.1 Continuous Distributions

1. Probability density
2. Change of scale
3. expectation and variance.

① sec 4.1 Probability density.

let X be a continuous RV

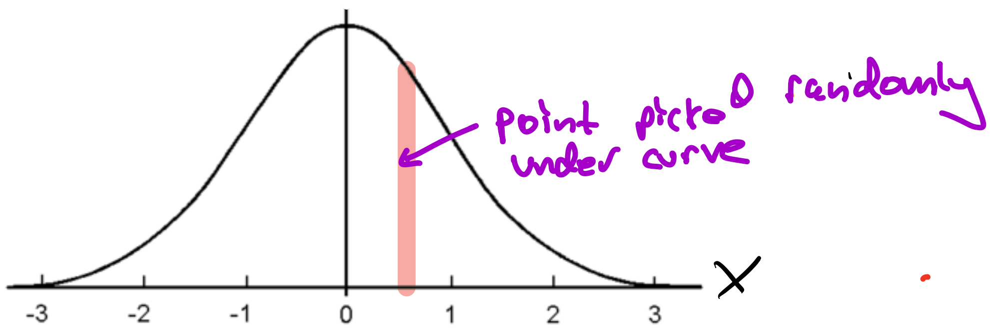
The probability density (histogram) of X is described by a prob densisity function

$$f(x) \geq 0 \text{ for } x \in X$$

and $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

ex The standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)$$

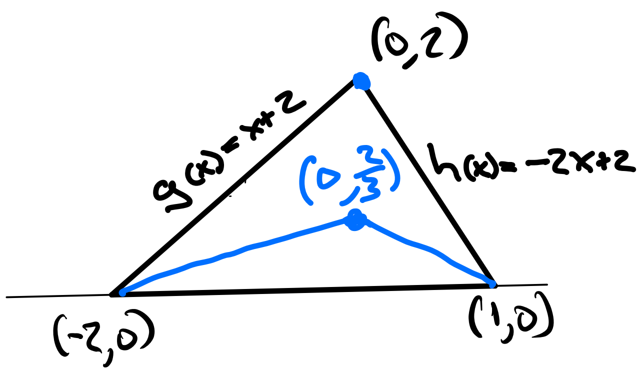


By geometry,
the chance of picking a pt under the curve in the red strip above is $f(x)dx$, where x is the x coordinate of the point, and dx is the width of the strip.

If $f(x)$ isn't a density, to make it a density divide it by the area under $f(x)$.

ex 4.1.12 b

Consider a point picked uniformly at random from the area inside the following triangle



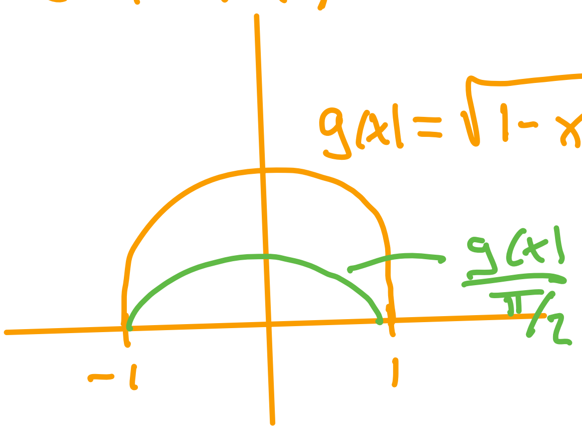
Find the density function of the x-coordinate $f(x)$

$A = \frac{1}{2}\cdot 3 \cdot 2 = \boxed{3}$

Pick a pt uniformly at random inside the triangle the chance you get the x coordinate of that point is the area of the strip above x divided by the total area,
we need to scale down the height of the triangle so the area of the triangle is 1. To do this divide $g(x), h(x)$ by $A=3$

$$f(x) = \begin{cases} \dfrac{x+2}{3} & -2 \le x \le 0 \\ \dfrac{-2x+2}{3} & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

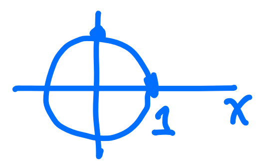
Note there is nothing special about the shape being a triangle, It could be a half circle with radius 1 for example,



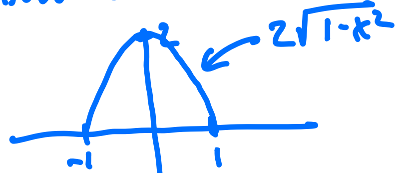
Here the area is $\frac{\pi}{2}$, To make g into a density divide it by $\pi/2$

$$f(x)=\frac{\sqrt{1-x^2}}{\pi/2}$$

Suppose the shape was a full circle of radius 1



Now part of the shape is under the x axis.
If you flip the bottom semicircle and add it to the top you get a shape

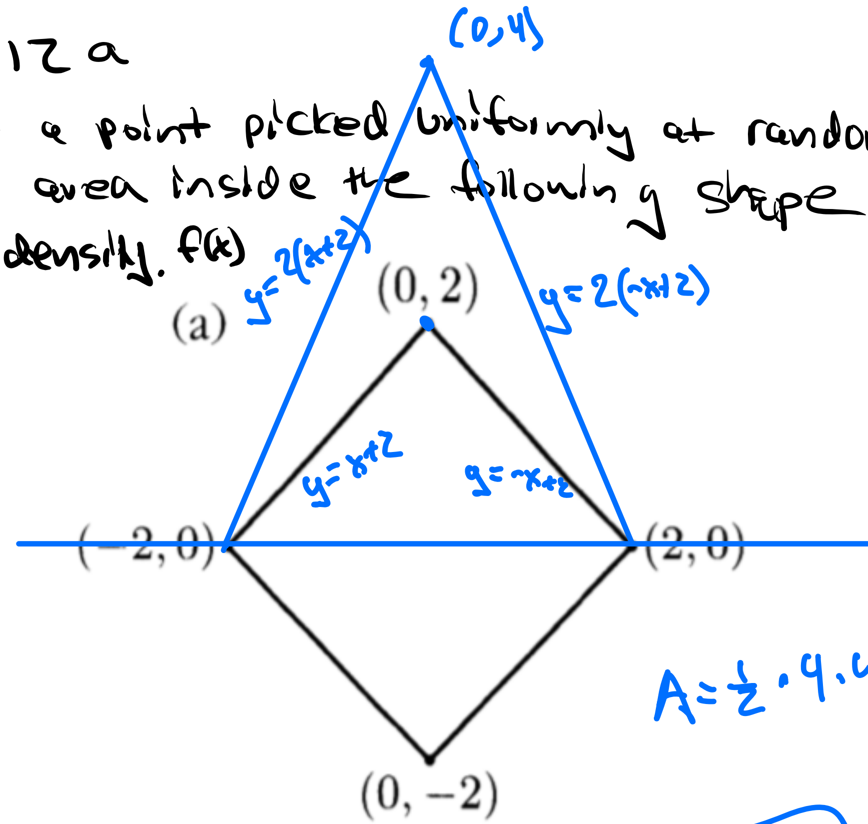


that is easier to think about. This isn't a density. To make it a density divide by π

$$f(x)=\begin{cases}\frac{2\sqrt{1-x^2}}{\pi} & -1\le x\le 1\\ 0 & \text{else}\end{cases}$$

ex 4.1.12 a

Consider a point picked uniformly at random from the area inside the following shape. Find the density $f(x)$.



$A = \frac{1}{2} \cdot 4 \cdot 4 = 8$

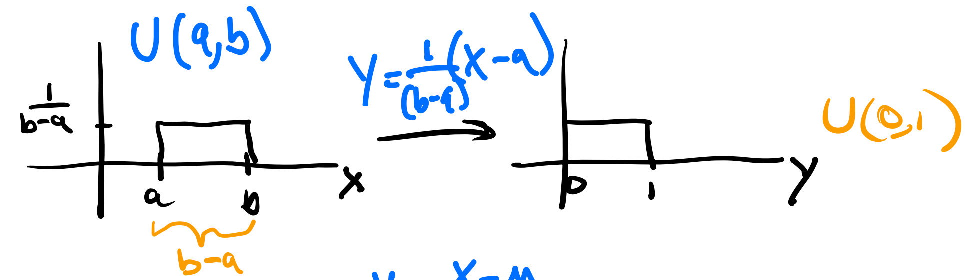
$$\boxed{f(x) = \begin{cases} \dfrac{2(x+2)}{8} & -2 \le x \le 0 \\ \dfrac{2(-x+2)}{8} & 0 \le x \le 2 \\ 0 & \text{else} \end{cases}}$$

② Change of scale

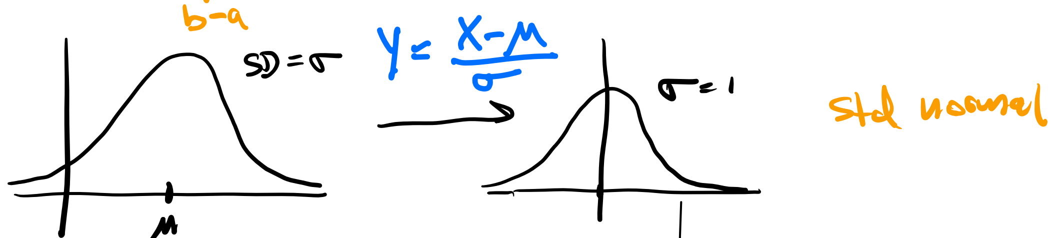
A <u>change of scale</u> is a transformation $Y = m + nX$, of X. The purpose is that it makes it easier to calculate $E(X)$ and $Var(X)$. It maps one density to another.

constants.

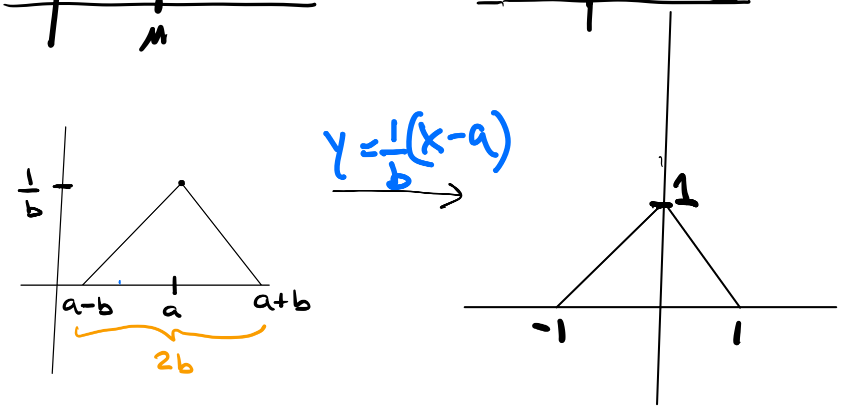
ex



ex



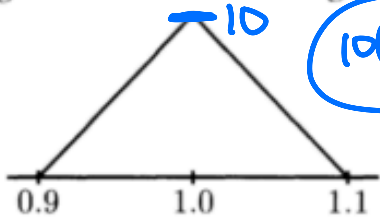
ex



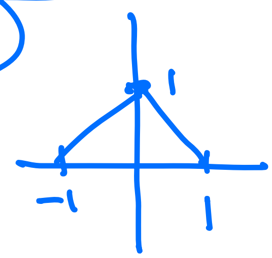
tinyurl.com/mav3-2023

Suppose a manufacturing process designed to produce rods of length 1 inch exactly, in fact produces rods with length distributed according to the density graphed below.

10



10(X-1)



You should change the scale of X= the length of rods to:

- a: X-1
- b: .1(X-1)
- c: 10X-1
- d: none of the above

③

Expectation and Variance

For discrete,

$$E(g(X)) = \sum_{x \in X} g(x) P(X=x)$$

For continuous,

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) P(X \in dx) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

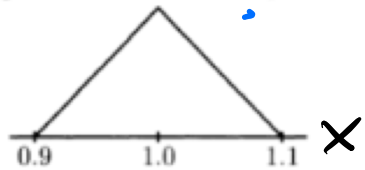
$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$Var(X) = E(X^2) - E(X)^2$$

See Wambo for example,

ex

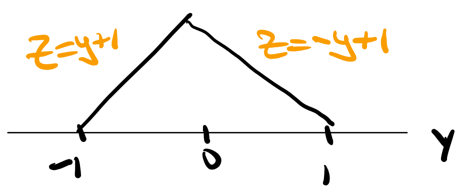
Suppose a manufacturing process designed to produce rods of length 1 inch exactly, in fact produces rods with length distributed according to the density graphed below.



Find the variance of the length of the rods.

$Y = 10(X-1)$ change of scale. easier to find.

$Var(Y) = 100\, Var(X) \Rightarrow Var(X) = \frac{Var(Y)}{100}$



Find the density of Y:

$$f(y) = \begin{cases} y+1 & -1 \le y \le 0 \\ -y+1 & 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

Find Var (X)

$$Var(Y) = E(Y^2) - (E(Y))^2$$

$$E(Y^2) = \int_{-1}^{0} y^2(y+1)dy + \int_{0}^{1} y^2(-y+1)dy$$

$$= \int_{-1}^{0} y^3 dy + \int_{-1}^{0} y^2 dy - \int_{0}^{1} y^3 dy + \int_{0}^{1} y^2 dy$$

$$= \frac{y^4}{4}\Big|_{-1}^{0} + \frac{y^3}{3}\Big|_{-1}^{0} - \frac{y^4}{4}\Big|_{0}^{1} + \frac{y^3}{3}\Big|_{0}^{1} = -\frac{1}{4} + \frac{1}{3} - \frac{1}{4} + \frac{1}{3}$$

$$= 2\left(\frac{1}{12}\right) = \boxed{\frac{1}{6}}$$

$E(Y) = 0 \Rightarrow Var(Y) = 1/6$

$$Var(X) = \frac{1/6}{100} = \boxed{1/600}$$